Physics of leakage of liquids into vessels

1996 J. Phys.: Condens. Matter 8 2775

(http://iopscience.iop.org/0953-8984/8/16/006)

J. Phys.: Condens. Matter **8** (1996) 2775–2780. Printed in the UK

Physics of leakage of liquids into vessels

D C Agrawal† and V J Menon‡

† Department of Farm Engineering, Institute of Agricultural Sciences, Banaras Hindu University, Varanasi-221005, India

‡ Department of Physics, Banaras Hindu University, Varanasi-221005, India

Received 27 June 1995, in final form 14 December 1995

Abstract. The practical importance of the leakage phenomenon of a liquid into a vessel is emphasized. The physics of the same is theoretically elucidated by incorporating the Young–Dupre relation, the Gibbs inequality, Poiseuille's flow, and the conditions for hydrostatic equilibrium.

1. Introduction

Capillarity phenomena are of great importance in physical, chemical, biological, and engineering sciences [1–4]. Of particular interest are the leakage of liquids into a vessel from outside, and the leakage in the reverse direction. Common examples are the supply of liquid fuels to automobile engines through the carburettor, the leakage of sea water into the hulls of vessels through minute cracks, passage of liquid metals into fine gaps for welding works, injection of lubricants into machine joints, sucking of ink through pores in blotting papers, forging of molten plastics into specific shapes by passage through fine pores, the cooling of water through the evaporation of leaked water in earthen pots kept in hot climates, the passage of sweat through the pores of human skin, the penetration of hot oil into foodstuffs during frying, the leakage of ink from fountain pens and ballpoints, etc. The physics of the leakage phenomenon into a vessel is somewhat different from that in the reverse direction and, in the present paper, we focus our attention on the former.

The basic physical principle involved in the process of leakage of a liquid into a vessel from outside is 'capillarity in a tube of insufficient length'. To be more specific, we recall that standard text-books [5, 6] quote the expression for the rise of a liquid in a capillary tube as

$$H_0 = a + 2T_{lv}\cos\theta_0/r\rho g \tag{1}$$

where H_0 is the equilibrium height of the risen column measured from the lower tip of the tube, a is the depth of immersion of the lower tip below the free surface, ρ the density of the liquid, r the inner radius of the tube, g the acceleration due to gravity, T_{lv} the surface tension of the liquid–vapour interface, and θ_0 ($< \pi/2$) is the Young–Dupre contact angle [7, 8]. It is assumed here that the tube is 'sufficiently long', i.e., $L > H_0$ where L is the actual length of the capillary tube. However, the case of a tube having 'insufficient length', i.e., $L < H_0$ (hereafter referred to as a short capillary), has not been treated in the literature [6, 9, 10] to the extent which it deserves. Not only is this problem of conceptual interest but it also has important practical applications as discussed in the first paragraph.

0953-8984/96/162775+06$19.50

In section 2 we recapitulate the Young–Dupre relation which defines the contact angle θ_0 at the junction of a solid, and a liquid and its vapour interfaces and also mention the Gibbs inequality [11, 12] obeyed by the contact angle when the triple line is pinned at a sharp edge. In section 3 we use the Poiseuille–Peiris–Tennakone [13] formula to calculate the instantaneous speed of the rising column. The energetics of motion of the liquid head as it overshoots the upper tip and its final equilibrium configuration, if any, will be discussed in section 4. In the same section we describe a nice application of these ideas to the leakage of water into a vessel whose bottom has developed a pin-hole crack. Since our emphasis in the present paper is on the applicational aspects of leakage, subtle conceptual/mathematical refinements on the dynamical features of capillarity will not be incorporated. Theses refinements arise due to heterogeneities of the solid surface leading to contact angle hysteresis [14] and the dependence of the contact angle on the velocity of the moving column [15].

2. Young–Dupre expression and Gibbs inequality

The formulation of the problem at hand may be conveniently made in terms of the interplay among the adhesive force A and the tensions T_{lv}, T_{sv} and T_{sl} of the liquid–vapour, solid–vapour and solid–liquid interfaces, respectively. For the sake of ready reference, let us recapitulate the conditions for these forces to be in equilibrium. Figure 1(*a*) shows these forces acting on a small element of the liquid situated at the junction of the three phases and having unit length perpendicular to the plane of the diagram. Resolving these forces parallel and perpendicular to the solid surface, one finds that the said element will be in equilibrium if

$$A = T_{lv} \sin\theta_0 \tag{2a}$$

$$T_{lv} \cos\theta_0 = T_{sv} - T_{sl} \tag{2b}$$

where θ_0 is the equilibrium contact angle. These are the celebrated Young–Dupre relations [8] and some of their well known properties are as follows. (2*a*) yields the value of the adhesive force while (2*b*) may be regarded as defining the equilibrium contact angle in terms of T_{sl}, T_{sv} and T_{lv}. The interfacial films are generally a few micrometers thick. Clearly, as long as the interfaces have a finite area the contact angle is acute, $\pi/2$, or obtuse when T_{sv} is greater than, equal to, or less than T_{sl}. Of course, in the case of the tap water–ordinary glass system θ_0 is acute, nearly 18° at room temperature [6].

Very near the upper tip of the tube, due to geometrical considerations, an exceptional case may arise because the three-phase contact line now meets a mathematically sharp solid edge. The equilibrium condition on the contact angle is expressed by the following inequality due to Gibbs [11, 12]:

$$\theta_0 \leqslant \theta \leqslant (180^\circ - \phi) + \theta_0 \tag{2c}$$

where ϕ is the angle subtended by the two surfaces forming the solid edge. In the case of the upper tip of our capillary tube $\phi = 90^\circ$, implying that, although the lower limit of the inequality remains acute, the upper limit becomes obtuse.

The methods of measuring T_{lv} are well known [5]; however, those of determining T_{sl} and T_{sv} are more subtle. For example, in the case of the mica–water–vapour system one employs the following cleavage technique [16]. The basic principle involved is the determination of the work done to cleave a thin strip of mica by measuring the forces required to maintain a given separations of the ends of the mica strip. High-resolution multiple-beam interference fringes are used to locate the line of bifuration of the mica sheets and, hence, to determine

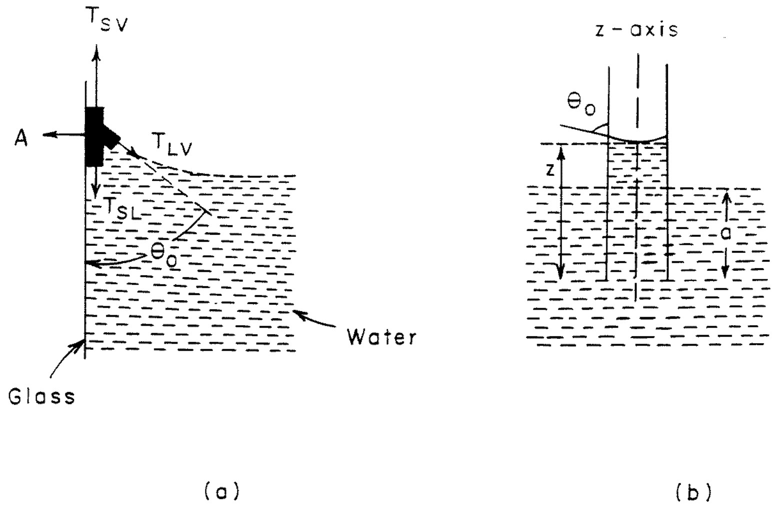

Figure 1. (*a*) A diagram showing the forces A, T_{lv}, T_{sv}, and T_{sl} acting on a small element of the fluid situated at the junction of the interfacial films. (*b*) A diagram showing a capillary tube dipped vertically into a liquid with its lower end at depth a below the free surface. The contact angle θ_0 and the time-dependent rise z are marked.

the area of new interface formed during the cleavage process. The apparatus, fitted with a sensitive hygrometer, is so arranged that both sample and cleavage mechanism can be completely surrounded by the liquid or vapour under examination. Hence, it becomes possible to measure solid–vapour and solid–liquid interfacial energies directly. Typical values of interfacial tensions for the case of water and hexane with mica as the solid are reported in table 1.

Table 1. Parameters in the Young equation (dyne cm^{-1}) at room temperature [16]. Here the solid (s) is mica.

Liquid	T_{sv}	T_{sl}	T_{lv}
Water	182.8	107.3	72.8
Hexane	271	255	18.4

Apart from the above-mentioned facts about the contact angle and interfacial tensions the final equilibrium configuration of the liquid head after crossing the upper tip also depends crucially on the speed of the rising column, to which we turn in the next section.

3. The speed of the rising column

Suppose a short capillary of length $L < H_0$ is dipped vertically in a liquid of viscosity η as shown in figure 1(b). Let z denote the height of the liquid column measured from the lower tip at time t, and $\dot{z} \equiv \mathrm{d}z/\mathrm{d}t$ denote its speed. A solution to the hydrodynamical problem of capillary motion under the simultaneous influence of surface tension, gravity, and viscosity was attempted by the present authors [17] in 1987 using the Newtons' law for a variable mass system coupled with an *ad hoc* assumption about the velocity gradient at the wall of the tube. However, since the assumption of the no-slip condition at the wall is more appropriate, one may prefer to employ the Poiseuille formula for the velocity as was done by Peiris and Tennakone [13]. In our notation their formula reads

$$\dot{z} = (\rho g H_0 r^2/8\eta)[1/z - 1/H_0]. \tag{3}$$

Furthermore, in Poiseuille's theory the kinetic energy when the liquid column reaches the top is given by

$$E_{kin} = \tfrac{1}{2}\pi r^2 L \dot{z}_{top}^2 \rho. \tag{4}$$

Of course, at $z = 0$ the formula (3) is inapplicable because the correct initial speed in accordance with the Torricelli theorem should be $(2gH_0)^{1/2}$. However, very soon the deceleration of the column becomes negligible so (3) starts operating. Numerically, for the case of the tap water–ordinary glass system the relevant typical parameters are as follows:

$$\begin{aligned} &\rho = 1\ \mathrm{g\ cm^{-3}} \qquad \theta_0 = 18^\circ \qquad \phi = 90^\circ \\ &r = 10^{-3}\ \mathrm{cm} \qquad \eta = 0.01\ \mathrm{Poise} \\ &L = 10\ \mathrm{cm} \qquad T = 70\ \mathrm{dyne\ cm^{-1}}. \end{aligned} \tag{5a}$$

The output Poiseuille–Peiris–Tennakone speeds $\dot{z}_{top}$ at the top of the tube become

$$\dot{z}_{top} = 0.16\ \mathrm{cm\ s^{-1}} \qquad \text{for } a = 3\ \mathrm{cm} \tag{5b}$$

$$\dot{z}_{top} = 0.18\ \mathrm{cm\ s^{-1}} \qquad \text{for } a = 20\ \mathrm{cm}. \tag{5c}$$

Clearly, $\dot{z}_{top}$ increases monotonically with the immersion depth a. The background developed in the above sections will be utilized in the actual theme of the present paper discussed in the next section.

4. Rise in a short capillary and its application

It is convenient to discuss two cases separately.

4.1. Case A: when a portion of the capillary is above the free surface ($L > a$)

For a tube dipped as shown in figure 2 the liquid will start at $z = 0$ with initial speed $(2gH_0)^{1/2}$ and will go on rising in accordance with (3) until the meniscus reaches the upper tip. Now, owing to the abrupt change of the geometry at the tip, several events start happening together in the following manner. As, due to inertia, the liquid head overshoots the tip the kinetic energy of the liquid is partially converted into the increased surface energy, remembering that the free liquid head will tend to acquire an approximately spherical-segment shape. The momentary change of the contact angle of the advancing liquid from an acute to an obtuse value is consistent with the range permitted by the Gibbs inequality (2*c*). Also, the solid–vapour interface disappears within the tube but reappears on the cross-section of the tube acting horizontally. Furthermore, the overshot segment corresponds to a metastable state because the surface has now acquired a reverse curvature [10] and hence the excess internal pressure will tend to push the liquid back down the tube. In addition, since all three forces, namely the surface tension, gravity, and viscous drag, have become downward the kinetic energy of the column drastically decreases. Finally, the kinetic energy (cf (4)) of whatsoever liquid remains within the tube will dissipate away through a couple of oscillations to yield a final static rise given by

$$L = a + 2T_{lv}\cos\theta_f/r\rho g \qquad \cos\theta_f > 0 \tag{6}$$

where θ_f is the final changed contact angle. This θ_f is still acute (though larger than θ_0) so it can support the apparent weight of the risen column in the static situation (see figure 2) in which the viscous force, of course, vanishes. Such a θ_f is also consistent with the Gibbs inequality (2*c*).

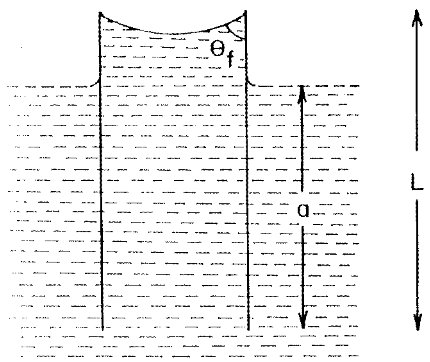

Figure 2. The final static equilibrium configuration in the case of a short capillary tube a portion of which protrudes outside the free surface. The liquid rises up to a point very close to the upper tip so the final contact angle θ_f becomes large but still acute.

4.2. Case B: when the upper tip of the capillary is below the free surface ($L < a$)

To be specific, let us consider the case of a vessel whose bottom has developed a pin-hole crack, immersed as shown in figure 3. The time-dependent motion of the liquid column leading to an initial spilling of the liquid is essentially the same as described in the previous case. Prior to achieving equilibrium the kinetic energy (cf (4)) of the liquid column is dissipated away in overcoming the viscous dissipation and storing the extra surface energy of the obtuse protrusion. These oscillations occur because of the change of the direction of the surface tension force from upward to downward and vice versa as mentioned above. Finally, the static configuration will be achieved with an obtuse contact angle θ_f so that the surface tension pull (now acting downwards) can balance the upward thrust due to buoyancy, yielding once again

$$L = a + 2T_{lv}\cos\theta_f/r\rho g \qquad \cos\theta_f < 0. \tag{7}$$

Of course, an obtuse final contact angle is consistent with the physical fact that the internal pressure developed owing to the convex surface is acting downwards. Furthermore, the Gibbs inequality (2*c*) is again satisfied.

If a were increased by immersing the vessel further, the force of buoyancy would also increase; hence to counter balance it the surface tension pull would have to be enhanced by making θ_f more obtuse. The maximum depth a_{max} up to which equilibrium can be maintained without leakage would be such that the protrusion acquires a limiting contact angle $\theta_f = (180^\circ - \phi) + \theta_0 = (90 + \theta_0)$ as permitted by Gibbs inequality (2*c*). Hence, from (7) we obtain the maximum immersion depth for non-leakage [9] as

$$a_{max} = L + 2T_{lv}\sin\theta_0/r\rho g. \tag{8}$$

This corresponds to an unstable situation because if it were disturbed by any means such as by immersing the vessel still further continuous leakage of water would start. In order to illustrate the above ideas numerically, let us suppose that the vessel with a pin-hole crack is floating in water characterized by the parameters of (5). Then $a_{max} = 54$ cm. In the end we may remark that the value of a_{max} is an increasing function of L and T_{lv} but a decreasing

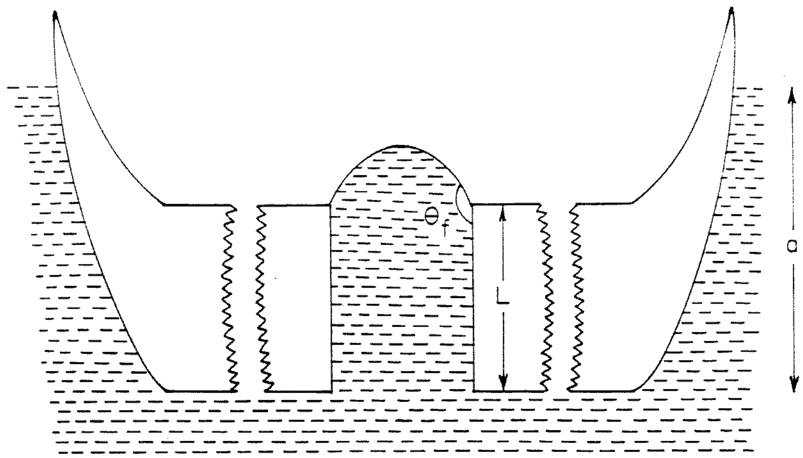

Figure 3. The final equilibrium rise within the pin-hole crack in the bottom of a vessel floating in water. The capillary tube is, therefore, lying wholly below the free surface. The meniscus now protrudes out at the upper tip, maintaining an obtuse contact angle.

function of r, ρ, and g. These dependences may be kept in mind while designing the pores as per requirements.

Acknowledgments

VJM thanks the UGC for the award of a Scientist B position.

References

[1] Rowlinson J S and Widom B 1982 *Molecular Theory of Capillarity* (Oxford: Oxford University Press) ch 1
[2] Jaycock M J and Parfitt G D 1983 *Chemistry of Interfaces* (New York: Ellis Horwood) pp 234–72
[3] Walker J 1977 *The Flying Circus of Physics with Answers* (New York: Wiley) physics problem 3.38, 3.100–3.107, 3.110
Cromer A H 1974 *Physics for the Life Sciences* (New York: McGraw-Hill) pp 180–7
[4] Lancaster J F 1984 *The Physics of Welding* (Oxford: Pergamon) pp 245–7
[5] Newman F H and Searle V H L 1951 *The General Properties of Matter* (London: Arnold) ch 6
[6] Mathur D S 1992 *Elements of Properties of Matter* (New Delhi: Chand) ch 14
[7] Sears F W and Zemansky M W 1965 *University Physics* (Reading, MA: Addison-Wesley) pp 302–4
[8] Defay R, Prigogine I, Bellemans A and Everett D H 1966 *Surface Tension and Adsorption* (London: Longmans) pp 14–16
[9] Problem No 4 of ch 6 in [5]. The answer quoted in this problem can be obtained from our (7) by the replacement $L \to 0$ and $\theta_f \to 180°$. This tacitly assumes the wall of the vessel to have negligible thickness and the final protrusion hemispherical shape [10].
[10] Marmur Abraham 1992 *Adv. Colloid Interface Sci.* **39** 13–33
[11] Gibbs J W 1906 *Scientific Papers* vol 1 Dover Reprint (New York: Dover) p 326
[12] Oliver J F, Huh C and Mason S G 1977 *J. Colloid Interface Sci.* **59** 568–81
[13] Peiris M G C and Tennakone K 1980 *Am. J. Phys.* **48** 415
[14] Leger L and Joanny J F 1992 *Rep. Prog. Phys.* **55** 431–86
[15] Tanner L H 1979 *J. Phys. D: Appl. Phys.* **12** 1473
[16] Bailey A I 1961 *J. Appl. Phys.* **32** 1407
Bailey A I and Kay Susan M 1967 *Proc. R. Soc.* A **301** 47
[17] Menon V J and Agrawal D C 1987 *Am. J. Phys.* **55** 63